Artist's conception of the 3-adic unit disk.

Drawing by A.T. Fomenko of Moscow State University, Moscow, U.S.S.R.

A (brief) introduction into p-adic analysis

2024 Directed Reading Program

Frank Connor Samy Lahlou

A New Norm(al)

What norms do we know over $\mathbb{Q}$?

- Our familiar norm: $\| \ \|_{\mathbb{R}}$ denoted $|\cdot|$ or $|\cdot|_\infty$
- The trivial norm: $\| \ \|_0$

The p norm: $|x|_p$

Let p be any prime number. For $a \in \mathbb{Q}$, $ord_p(a) :=$ the greatest $m \in \mathbb{Z}$ s.t. $a \equiv 0 \bmod (p^m)$.

$$|x|_p = \left\{ \begin{array}{ll} (1/p)^{ord_p(x)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{array} \right\} \quad (1)$$

Ostrowski's Theorem

Ostrowski's Theorem [Ost. 1916]

Every non-trivial norm on $\mathbb{Q}$ is equivalent to $| \ |_p$ for some prime p or for p = ∞.

$\mathbb{Q} \to \mathbb{Q}_p$

- $\forall p \;\; \mathbb{Q}$ is not complete with respect to $| \;\; |_p$.[1]
- Just like we can complete $\mathbb{Q}$ to $\mathbb{R}$ with respect to $| \cdot |$. We can complete $\mathbb{Q}$ with respect to $| \;\; |_p$. We call this new field $\mathbb{Q}_p$

> $\mathbb{Q}_p$
>
> The fields formed by the completions of $\mathbb{Q}$ with respect to $| \;\; |_p$

[1] Indeed, consider for some $a \in \{2, 3, \cdots, p-1\}$ $(x_n) := (a^{p^n})$) which is Cauchy but not convergent.

What is in $\mathbb{Q}_p$?

Take $x \in \mathbb{Q}_p$ for some prime p.

One non-canonical way to represent x is in the following form:

$$x = \sum_{k=l}^{\infty} a_k p^k \qquad \forall k \;\; a_k \in \mathbb{Z}/p\mathbb{Z} \;\; l \in \mathbb{Z}$$

Examples for p = 3

$$17 = 1 * 3^2 + 2 * 3^1 + 2 * 3^0$$

$$-\frac{1}{2} = \sum_{i=0}^{\infty} 3^i$$

The Weird Topology of the Metric Space $\mathbb{Q}_p$

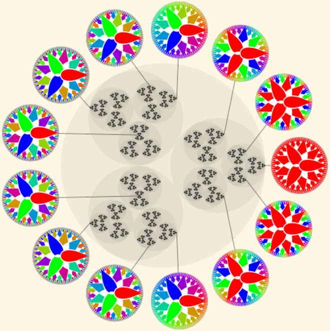

Figure: A representation of $\mathbb{Q}_3$ by Mel Choir (2018)

- $\mathbb{Q}_p$ is an ultra-metric space.
- $\mathbb{Q}_p$ is a totally disconnected topological space

Locally Constant Functions and P-adic Distributions

Locally constant function

A function $f : X \to Y$ is called locally constant on X if every point $x \in X$ has a neighborhood U s.t. $|f(U)| = 1$

P-adic distribution

[a] A p-adic distribution μ on X is a $\mathbb{Q}_p$-linear vector space homomorphism $\mathbb{Q}_p$-vector space of locally constant functions on X to $\mathbb{Q}_p$. If $f : X \to \mathbb{Q}_p$ is locally constant, instead of writing $\mu(f)$ we write $\int f\mu$

[a] If this is confusing just keep in mind step functions in $\mathbb{R}$

Analysis on $\mathbb{Q}_p$

- A distribution which is bounded on a compact set (for instance $\mathbb{Z}_p$) is called a measure.
- With measures we can do analysis, specifically we can:
  - Take Riemann sums
  - Define a p-adic logarithim
  - Define a p-adic gamma function
  - Work with L functions such as the Riemannn-Zeta function

A formula for $\zeta(2k)$

$$\zeta(2k) = (-1)^k \pi^{2k} \frac{2^{2k-1}}{(2k-1)!} \left(\frac{-B_{2k}}{2k} \right)$$

Beyond $\mathbb{Q}_p$: $\mathbb{C}_p$

If $\mathbb{Q}_p$ is a p-adic version of $\mathbb{R}$ what is the p adic analogy of $\mathbb{C}$?

Just like $\mathbb{R}$, $\mathbb{Q}_p$ is not algebraically closed. However, while $\mathbb{R}$ has one proper algebraic extension, $\mathbb{C}$, $\mathbb{Q}_p$ has an infinite degree extension.

> $\overline{\mathbb{Q}_p}$
>
> The infinite degree algebraic extension of $\mathbb{Q}_p$ s.t. it is closed.

However, we lost completeness in the process.

> $\mathbb{C}_p$
>
> The completion of $\overline{\mathbb{Q}_p}$ denoted as $\mathbb{C}_p$, is the smallest algebraically closed and complete field which contains $\mathbb{Q}_p$.

$\mathbb{C}_p$ is isomorphic to $\mathbb{C}$ as a ring.

References

- N. Koblitz, *p-adic Numbers, p-adic Analysis, and Zeta Functions*, New York Heidelberg Berlin, 1984.
- F. Gouvea, p-adic Numbers, Springer Science and2013 Business Media, 2013.