Solutions to Representation Theory of Finite Groups - Benjamin Steinberg

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Preface

The goal of this document is to share my personal solutions to the exercises of Representation Theory of Finite Groups by Benjamin Steinberg during my reading.

As a disclaimer, the solutions are not unique and there will probably be better or more optimized solutions than mine. Feel free to correct me or ask me anything about the content of this document at the following address : samy.lahloukamal@mcgill.ca

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Chapter 1

Introduction

[No exercises in this chapter.]

Chapter 2

Review of Linear Algebra

Exercise 2.1
Suppose that $A, B \in M_n(\mathbb{C})$ are commuting matrices, i.e., $AB = BA$. Let V_λ be an eigenspace of A. Show that V_λ is B-invariant.

Solution
Recall that

$$V_\lambda = \{v \in \mathbb{C}^n : Av = \lambda v\}$$

where $\lambda \in \mathbb{C}$ satisfies $Au = \lambda u$ for some $u \in \mathbb{C}^n \setminus \{0\}$. Let $v \in V_\lambda$ and let's show that $Bv \in V_\lambda$. Notice that

$$\begin{aligned} A(Bv) &= BAv \\ &= B\lambda v \\ &= \lambda(Bv) \end{aligned}$$

Thus, by definition, $Bv \in V_\lambda$. It follows that V_λ is B-invariant.

Exercise 2.2
Let V be an n-dimensional vector space and B a basis. Prove that the map $F : \mathrm{End}(V) \to M_n(\mathbb{C})$ given by $F(T) = [T]_B$ is an isomorphism of unital rings.

Solution
Recall that a unital ring is simply a ring with a multiplicative identity. Moreover, denote the elements in B by b_1, b_2, ..., b_n. Let's first show that F is a ring homomorphism, i.e., F preserves addition, multiplication (composition in $\mathrm{End}(V)$) and sends the identity transformation to the identity matrix.

- (Preserves Addition) Let $T_1, T_2 \in \mathrm{End}(V)$ and consider the matrices $[T_1]_B$, $[T_2]_B$ and $[T_1 + T_2]_B$. For all $j \in \{1, ..., n\}$, we can write

  $$T_1 b_j = \sum_{i=1}^{n} \alpha_{ij} b_i \qquad \text{and} \qquad T_2 b_j = \sum_{i=1}^{n} \beta_{ij} b_i$$

  for some scalars $\alpha_{ij}, \beta_{ij} \in \mathbb{C}$ with $1 \leq i, j \leq n$. It follows by definition that $[T_1]_B = (\alpha_{ij})$ and $[T_2]_B = (\beta_{ij})$. Moreover, since for all $j \in \{1, ..., n\}$ we have

  $$(T_1 + T_2)b_j = \sum_{i=1}^{n} \alpha_{ij} b_i + \sum_{i=1}^{n} \beta_{ij} b_i = \sum_{i=1}^{n} (\alpha_{ij} + \beta_{ij}) b_i$$

then we get

$$\begin{aligned} F(T_1+T_2) &= [T_1+T_2]_B \\ &= (\alpha_{ij}+\beta_{ij}) \\ &= (\alpha_{ij})+(\beta_{ij}) \\ &= F(T_1)+F(T_2) \end{aligned}$$

which proves that F preserves addition.

- (Preserves Multiplication) Let $T_1, T_2 \in \text{End}(V)$. Recall that multiplication in $\text{End}(V)$ is defined as the composition of functions, and multiplication in $M_n(\mathbb{C})$ is defined by the following formula:

$$(a_{ij}) \times (b_{ij}) = \left(\sum_{k=1}^{n} a_{ik}b_{kj}\right)$$

For all $j \in \{1, ..., n\}$, we can write

$$T_1 b_j = \sum_{i=1}^{n} \alpha_{ij} b_i \qquad \text{and} \qquad T_2 b_j = \sum_{i=1}^{n} \beta_{ij} b_i$$

for some scalars $\alpha_{ij}, \beta_{ij} \in \mathbb{C}$ with $1 \le i, j \le n$. It follows by definition that $[T_1]_B = (\alpha_{ij})$ and $[T_2]_B = (\beta_{ij})$. Moreover, since for all $j \in \{1, ..., n\}$ we have

$$\begin{aligned} (T_1 \circ T_2)b_j &= T_1(T_2 b_j) \\ &= T_1 \sum_{k=1}^{n} \beta_{kj} b_k \\ &= \sum_{k=1}^{n} \beta_{kj} T_1 b_k \\ &= \sum_{k=1}^{n} \beta_{kj} \left(\sum_{i=1}^{n} \alpha_{ik} b_i\right) \\ &= \sum_{k=1}^{n}\sum_{i=1}^{n} \beta_{kj}\alpha_{ik} b_i \\ &= \sum_{i=1}^{n} \left(\sum_{k=1}^{n} \alpha_{ik}\beta_{kj}\right) b_i \end{aligned}$$

It follows that $[T_1 \circ T_2]_B = (\sum_{k=1}^{n} \alpha_{ik}\beta_{kj})$. Therefore, we get

$$\begin{aligned} F(T_1T_2) &= [T_1 \circ T_2]_B \\ &= \left(\sum_{k=1}^{n} \alpha_{ik}\beta_{kj}\right) \\ &= (\alpha_{ij})(\beta_{ij}) \\ &= [T_1]_B[T_2]_B \\ &= F(T_1)F(T_2) \end{aligned}$$

Therefore, F preserves the multiplication.

- Consider now the identity map $\mathrm{id}_V : V \to V$ which is the multiplicative identity in $\mathrm{End}(V)$. Let's show that $F(\mathrm{id}_V) = I_n$. To do so, notice that for all $j \in \{1, ..., n\}$, we have

$$\mathrm{id}_V b_j = b_j = \sum_{i=1}^{n} \alpha_{ij} b_i$$

  where $\alpha_{ij} = 1$ when $i = j$ and $\alpha_{ij} = 0$ otherwise. It follows that $[\mathrm{id}_V]_B = (\alpha_{ij})$. Therefore, the matrix $[\mathrm{id}_V]_B$ is equal to zero for all of its entries except on the diagonal where it is equal to one. It follows that

$$F(\mathrm{id}_V) = [\mathrm{id}_V]_B = I_n$$

Now that we showed that F is a ring homomorphism, we need to show that it is also a bijection:

- (Injectivity) Let T_1 and T_2 be linear maps from V to V such that $F(T_1) = F(T_2)$, then $[T_1]_B = [T_2]_B$. Recall $[T_1]_B$ and $[T_2]_B$ are defined as (α_{ij}) and $(\beta_{ij})_{ij}$ respectively where

$$T_1 b_j = \sum_{i=1}^{n} \alpha_{ij} b_i \qquad \text{and} \qquad T_2 b_j = \sum_{i=1}^{n} \beta_{ij} b_i$$

  for all $i, j \in \{1, ..., n\}$. Since $[T_1]_B = [T_2]_B$, then $\alpha_{ij} = \beta_{ij}$ for all $i, j \in \{1, ..., n\}$. It follows that $T_1 b_i = T_2 b_i$ for all $i \in \{1, ..., n\}$. But since any function from B to V can be uniquely extended to a linear map from V to V, then $T_1 = T_2$. Therefore, F is injective.

- (Surjectivity) Let $(\alpha_{ij}) \in M_n(\mathbb{C})$ and consider the map defined by

$$T b_j = \sum_{i=1}^{n} \alpha_{ij} b_i$$

  for all $i, j \in \{1, ..., n\}$. Since any map from B to V can be uniquely extended to a linear map from V to V, then $T \in \mathrm{End}(V)$. Moreover, by construction, $F(T) = [T]_B = (\alpha_{ij})_{ij}$. Therefore, F is surjective.

Since F is a bijective ring homomorphism, then F is a ring isomorphism.

Exercise 2.3
Let V be an inner product space and let $W \leq V$ be a subspace. Let $v \in V$ and define $\hat{v} \in W$ as in the proof of Proposition 2.2.3. Prove that if $w \in W$ with $w \neq \hat{v}$, then $\|v - \hat{v}\| < \|v - w\|$. Deduce that $\hat{v}$ is independent of the choice of orthonormal basis for W. It is called the *orthonormal projection* of v onto W.

Solution
TODO

Exercise 2.4
Prove that $(AB)^* = B^* A^*$.

Solution
If we write $A = (a_{ij})$ and $B = (b_{ij})$, then

$$AB = (a_{ij})(b_{ij}) = \left(\sum_{k=1}^{n} a_{ik} b_{kj} \right)$$

It follows that

$$(AB)^* = \left(\sum_{k=1}^{n} a_{ik} b_{kj} \right)^* = \left(\sum_{k=1}^{n} \overline{a_{jk}} \cdot \overline{b_{ki}} \right)$$

Similarly, since $A^* = (\overline{a_{ji}})$ and $B^* = (\overline{b_{ji}})$, then

$$B^* A^* = (\overline{a_{ji}})(\overline{b_{ji}}) = \left(\sum_{k=1}^{n} \overline{b_{ki}} \cdot \overline{a_{jk}} \right)$$

Therefore, combining the last two results gives us $(AB)^* = B^* A^*$.

Exercise 2.5
Prove that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

Solution
If we write $A = (a_{ij})_{ij}$ and $B = (b_{ij})_{ij}$, then

$$AB = (a_{ij})(b_{ij}) = \left(\sum_{k=1}^{n} a_{ik} b_{kj} \right)$$

It follows that

$$\mathrm{Tr}(AB) = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki} = \sum_{k=1}^{n} \sum_{i=1}^{n} b_{ki} a_{ik}$$

Since the variables i and k are just dummy variables, then we can interchange the variable names without changing the value of the sum (replace the i's by k's and the k's by i's). From this, we get:

$$\mathrm{Tr}(AB) = \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ik} a_{ki}$$

Similarly, since we have

$$BA = (b_{ij})(a_{ij}) = \left(\sum_{k=1}^{n} b_{ik} a_{kj} \right)$$

then

$$\mathrm{Tr}(BA) = \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ik} a_{ki}$$

Therefore, $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

Exercise 2.6
TODO

Exercise 2.7
TODO

Exercise 2.8
TODO

Exercise 2.9
TODO

Chapter 3

Basic Definitions and First Examples

Exercise 3.1
Let $\varphi : D_4 \to \mathrm{GL}_2(\mathbb{C})$ be the representation given by

$$\varphi(r^k) = \begin{bmatrix} i^k & 0 \\ 0 & (-i)^k \end{bmatrix} \ , \ \varphi(sr^k) = \begin{bmatrix} 0 & (-i)^k \\ i^k & 0 \end{bmatrix}$$

where r is rotation counterclocwise by $\pi/2$ and s is reflection over the x-axis. Prove that φ is irreducible.

Solution
By Proposition 3.1.19, it suffices to show that there is no common eigenvector to all matrices in the image of φ. In particular, it suffices to show that

$$A = \varphi(r) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad \text{and} \quad B = \varphi(sr) = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = B$$

have no common eigenvector. Let's compute the characteristic polynomial of B:

$$\begin{aligned} p_B(x) &= \det(xI_2 - B) \\ &= \det\left(\begin{bmatrix} x & i \\ -i & x \end{bmatrix}\right) \\ &= x^2 - 1 \\ &= (x-1)(x+1) \end{aligned}$$

It follows that the eigenvalues of B are precisely 1 and -1. Let's compute their respective eigenspaces:

$$\begin{aligned} V_1 &= \{v \in \mathbb{C}^2 : Bv = v\} \\ &= \{(v_1, v_2) \in \mathbb{C}^2 : \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\} \\ &= \{(v_1, v_2) \in \mathbb{C}^2 : v_1 = -iv_2 \ \text{ and } \ v_2 = iv_1\} \\ &= \{(v_1, v_2) \in \mathbb{C}^2 : v_1 = -iv_2\} \\ &= \mathbb{C}\begin{bmatrix} -i \\ 1 \end{bmatrix} \end{aligned}$$

Similarly, we find

$$V_{-1} = \mathbb{C}\begin{bmatrix} i \\ 1 \end{bmatrix}$$

Hence, it suffices to show that $\begin{bmatrix} -i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} i \\ 1 \end{bmatrix}$ are not eigenvectors of A. To do so, notice that

$$A \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix} \notin \mathbb{C} \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

and

$$A \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -i \end{bmatrix} \notin \mathbb{C} \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Thus, both vectors are not eigenvectors of A. Therefore, φ must be irreducible.

Exercise 3.2

Let $\varphi : G \to GL(V)$ be equivalent to an irreducible representation. Then φ is irreducible.

Solution

Let $\psi : G \to GL(W)$ be an irreducible representation of G such that $\varphi \sim \psi$, then there exists a vector space isomorphism $T : V \to W$ such that $T\varphi_g = \psi_g T$ for all $g \in G$. Suppose by contradiction that φ is not an irreducible representation, then there exists a proper G-invariant subspace $V_0 \leq V$ different than $\{0\}$.

Consider the set $W_0 = \{Tv : v \in V_0\}$. Let's prove that W_0 is a subspace of W. By linearity, for all $\alpha, \beta \in \mathbb{C}$ and $v, w \in V_0$, we have

$$\alpha Tv + \beta Tw = T(\alpha v + \beta w)$$

Since V_0 is a subspace, then it is closed under linear combinations. Hence, $\alpha v + \beta w \in V_0$. It follows that $\alpha Tv + \beta Tw \in W_0$. Thus, W_0 is a subspace of W.

Let's show that $W_0 \neq \{0\}$. Since $V_0 \neq \{0\}$, then there exists a non-zero vector $v \in V_0$. It follows that $Tv \in W_0$. Since T is an isomorphism, then Tv is non-zero as well. Thus, $W_0 \neq \{0\}$.

Let's show that W_0 is a proper subspace of W. Since V_0 is a proper subspace of V, then there is a vector $v \in V$ such that $v \notin V_0$. Consider the vector $w = Tv \in W$. If $w \in V_0$, then there exists a $v_0 \in V_0$ such that $w = Tv_0$. Hence, $Tv = Tv_0$. By injectivity of T, there $v = v_0 \in V_0$. A contradiction. Hence, $w \notin W$. Thus, W_0 is a proper subspace of W.

Finally, let's show that W_0 is G-invariant. Let $g \in G$ and $w \in W_0$, then there is a $v \in V_0$ such that $w = Tv$. Moreover, since V_0 is G-invariant, then $\varphi_g v \in V_0$. Thus,

$$\psi_g w = \psi_g Tv = T\varphi_g v \in W_0$$

Hence, W_0 is G-invariant.

However, this is a contradiction because what we showed is that W has a proper G-invariant subspace $W_0 \neq \{0\}$. This contradicts the fact that ψ is irreducible. Therefore, φ is irreducible.

Exercise 3.3

Let $\varphi, \psi : G \to \mathbb{C}^*$ be one-dimensional representations. Show that φ is equivalent to ψ if and only if $\varphi = \psi$.

Solution
($\Longleftarrow$) Suppose that $\varphi = \psi$ and consider the identity map $T : \mathbb{C} \to \mathbb{C}$. Since T is a vector space isomorphism and

$$T\varphi_g = \psi_g T$$

for all $g \in G$, then it follows that φ is equivalent to ψ.
($\Longrightarrow$) Suppose that φ is equivalent to ψ, then there exists a vector space isomorphism $T : \mathbb{C} \to \mathbb{C}$ such that

$$T\varphi_g = \psi_g T$$

for all $g \in G$. However, notice that any isomorphism from $\mathbb{C}$ to $\mathbb{C}$ is simply a multiplication by a scalar. To understand why, let $\alpha = T(1)$ and let $x \in \mathbb{C}$, then

$$T(x) = T(x \cdot 1) = xT(1) = \alpha x$$

Moreover, α must be non-zero by injectivity of T. Hence, by commutativity in $\mathbb{C}$, given a $g \in G$, we get

$$\begin{aligned} T\varphi_g = \psi_g T &= \alpha\varphi_g = \psi_g \alpha \\ &= \alpha\varphi_g = \alpha\psi_g \\ &= \varphi_g = \psi_g \end{aligned}$$

Since it holds for all $g \in G$, then $\varphi = \psi$.

Exercise 3.4
Let $\varphi : G \to \mathbb{C}^*$ be a representation. Suppose that $g \in G$ has order n.

1. Show that $\varphi(g)$ is an nth-root of unity (i.e., a solution to the equation $z^n = 1$).

2. Construct n inequivalent one-dimensional representations $\mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^*$.

3. Explain why your representations are the only possible one-dimensional representations.

Solution

1. Since φ is a group homomorphism, then the identity 1_G in g is mapped to $1 \in \mathbb{C}^*$. Moreover, we can show by induction on k that

   $$\varphi(g^k) = \varphi(g)^k$$

   for all $k \in \mathbb{Z}$. By plugging-in $k = n$, we get

   $$\varphi(g)^n = \varphi(g^n) = \varphi(1_G) = 1$$

   Therefore, $\varphi(g)$ is a nth-root of unity.

2. Consider the mappings $\varphi_k : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^*$ defined by

   $$\varphi_k([m]) = e^{2\pi imk/n}$$

where $k = 1, ..., n$. First, let's show that each φ_k is well-defined. Let $k \in \{1, ..., n\}$ and let $[m_1], [m_2] \in \mathbb{Z}/n\mathbb{Z}$ such that $[m_1] = [m_2]$, then there exists a $t \in \mathbb{Z}$ such that $m_2 = m_1 + tn$. It follows that

$$\begin{aligned}\varphi_k([m_2]) &= e^{2\pi i m_2 k/n} \\ &= e^{2\pi i(m_1+tn)k/n} \\ &= e^{(2\pi i m_1 k/n)+(2\pi i tnk/n)} \\ &= e^{2\pi i m_1 k/n} \cdot e^{2\pi i tk} \\ &= \varphi_k([m_1])\end{aligned}$$

Therefore, the mappings are all well-defined. Let's now show that each mapping is a representation by showing that it is a homomorphism. Let $k \in \{1, ..., n\}$ and $[m_1], [m_2] \in \mathbb{Z}/n\mathbb{Z}$, then

$$\begin{aligned}\varphi_k([m_1] + [m_2]) &= \varphi_k([m_1 + m_2]) \\ &= e^{2\pi i(m_1+m_2)k/n} \\ &= e^{(2\pi i m_1 k/n)+(2\pi i m_2 k/n)} \\ &= e^{2\pi i m_1 k/n} e^{2\pi i m_2 k/n} \\ &= \varphi_k([m_1])\varphi_k([m_2])\end{aligned}$$

Therefore, each φ_k is a representation. To show that these n representations are inequivalent, recall that they are all distinct since they all map [1] to a different element. Using exercise 3.3, it directly follows that they are all inequivalent since they are not strictly equal.

3. First, recall that any one-dimensional representation of G is equivalent to a representation $\varphi : G \to \mathbb{C}^*$. Hence, it suffices to only consider the representations $\varphi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^*$. Let $\varphi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^*$ be an arbitrary representation of $\mathbb{Z}/n\mathbb{Z}$. Since $[1] \in \mathbb{Z}/n\mathbb{Z}$ has order n, then by part 1., $\varphi([1])$ must be a nth root of unity. Hence, there exists a $k \in \{1, ..., n\}$ such that

$$\varphi([1]) = e^{2\pi i k/n}$$

From this, since φ is a group homomorphism, then we can deduce that for all $[m] \in \mathbb{Z}/n\mathbb{Z}$, we have

$$\begin{aligned}\varphi([m]) &= \varphi(m \cdot [1]) \\ &= \varphi([1])^m \\ &= (e^{2\pi i k/n})^m \\ &= e^{2\pi i mk/n} \\ &= \varphi_k([m])\end{aligned}$$

Since it holds for all $[m] \in \mathbb{Z}/n\mathbb{Z}$, then $\varphi = \varphi_k$. Therefore, the representations described in previous part are the only possible one-dimensional representations of $\mathbb{Z}/n\mathbb{Z}$.

Exercise 3.5
Let $\varphi : G \to GL(V)$ be a representation of a finite group G. Define the *fixed subspace*

$$V^G = \{v \in V | \varphi_g v = v, \forall g \in G\}.$$

1. Show that V^G is a G-invariant subspace.

2. Show that
$$\frac{1}{|G|}\sum_{h \in G} \varphi_h v \in V^G$$
for all $v \in V$.

3. Show that if $v \in V^G$, then
$$\frac{1}{|G|}\sum_{h \in G} \varphi_h v = v.$$

4. Conclude $\dim V^G$ is the rank of the operator
$$P = \frac{1}{|G|}\sum_{h \in G} \varphi_h.$$

5. Show that $P^2 = P$.

6. Conclude $\mathrm{Tr}(P)$ is the rank of P.

7. Conclude
$$\dim V^G = \frac{1}{|G|}\sum_{h \in G} \mathrm{Tr}(\varphi_g).$$

Solution

1. First, for completeness, let's prove that V^G is a subspace of V. It is non-empty because the zero vector is fixed by every linear map on V. Moreover, given any $\alpha, \beta \in \mathbb{C}$, $u, v \in V^G$ and $g \in G$, we get
$$\varphi_g(\alpha u + \beta v) = \alpha\varphi_g u + \beta\varphi_g v = \alpha u + \beta v$$
which shows that $\alpha u + \beta v \in V^G$. Thus, V^G is a subspace since it is a non-empty subset of V that it closed under linear combinations.
Now, simply notice that by definition, for any $v \in V^G$ and $g \in G$, we have $\varphi_g v = v$. Thus, V^G is G-invariant.

2. Let $v \in V$ and consider the element $x = \frac{1}{|G|}\sum_{h \in G} \varphi_h v \in V$. To show that $x \in V^G$, let $g \in G$ be arbitrary and let's show that $\varphi_g x = x$. By linearity of

φ_g and using the fact that φ is a homomorphism, we get

$$\begin{aligned}\varphi_g x &= \varphi_g \frac{1}{|G|}\sum_{h\in G}\varphi_h v \\ &= \frac{1}{|G|}\sum_{h\in G}\varphi_g\varphi_h v \\ &= \frac{1}{|G|}\sum_{h\in G}\varphi_{gh} v\end{aligned}$$

Notice that the sum is taken over $h \in G$ but the only time it is used is in the subscript φ_{gh}. Since the function $h \mapsto gh$ is a bijection from G to G and the sum is finite, then this sum is simply a rearrangement of the sum in which we replace gh by h. Hence,

$$\begin{aligned}\varphi_g x &= \frac{1}{|G|}\sum_{h\in G}\varphi_{gh} v \\ &= \frac{1}{|G|}\sum_{h\in G}\varphi_h v \\ &= x\end{aligned}$$

Since it holds for all $g \in G$, then $x \in V^G$.

3. To do so, let $v \in V^G$ and recall that by definition, $\varphi_h v = v$ for all $h \in G$:

$$\frac{1}{|G|}\sum_{h\in G}\varphi_h v = \frac{1}{|G|}\sum_{h\in G} v = \frac{1}{|G|}|G|v = v$$

which proves the desired formula.

4. Define the operator

$$P = \frac{1}{|G|}\sum_{h\in G}\varphi_h$$

We already now from part 2. of this question that $\text{Im}(P) \subset V^G$. Moreover, we know from part 3. of this question that any element of V^G is in the image of P since $v = Pv$ for all $v \in V^G$. Thus, $\text{Im}(P) = V^G$. It follows that the rank of P is $\dim V^G$.

5. First, since the image of P is V^G, then

$$P^2 = P|_{V^G} \circ P$$

But we already know that P acts as the identity map on V^G. In other words: $P|_{V^G} = id_{V^G}$. Therefore,

$$P^2 = P|_{V^G} \circ P = id_{V^G} \circ P = P$$

6. Let $B = \{b_1, b_2, ..., b_n\}$ be a basis for V^G and extend it to a basis $B' = \{b_1, ..., b_n, b'_1, ..., b'_m\}$ of V where $n = \dim V^G$ and $n + m = \dim V$. Consider the matrix representation $M = [P]_{B'}$ of P in the basis B'. Notice that the first n columns of the matrix are simply the vectors $e_i \in \mathbb{C}^n$ where $1 \leq i \leq n$ since P acts as the identity on V^G. If we write $M = (m_{ij})_{1\leq i,j\leq n+m}$, then the last sentence implies that $m_{ii} = 1$ for all $1 \leq i \leq n$.
   Moreover, notice that $Pb'_i \in V^G$, hence, its representation in the B' basis only involves the vectors $\{b_1, ..., b_n\}$. Again, this translates to $m_{ii} = 0$ for all $n + 1 \leq i \leq n + m$.
   Therefore, since the trace of a transformation is the trace of any of its matrix representation, then

$$\begin{aligned}\mathrm{Tr}(P) &= \mathrm{Tr}(M)\\ &= \sum_{i=1}^{n+m} m_{ii}\\ &= \sum_{i=1}^{n} m_{ii} + \sum_{i=n+1}^{n+m} m_{ii}\\ &= \sum_{i=1}^{n} 1 + \sum_{i=n+1}^{n+m} 0\\ &= n + 0\\ &= \dim V^G\end{aligned}$$

7. Since the trace is linear, then

$$\mathrm{Tr}(P) = \mathrm{Tr}\left(\frac{1}{|G|}\sum_{h\in G}\varphi_h\right) = \frac{1}{|G|}\sum_{h\in G}\mathrm{Tr}(\varphi_h)$$

   which is the desired result.

Exercise 3.6
Let $\varphi : G \to GL_n(\mathbb{C})$ be a representation.

1. Show that setting $\psi_g = \overline{\varphi_g}$ provides a representation $\psi : G \to GL_n(\mathbb{C})$. It is called the *conjugate representation*. Give an example showing that φ and ψ do not have to be equivalent.

2. Let $\chi : G \to \mathbb{C}^*$ be a degree 1 representation of G. Define a map $\varphi^\chi : G \to GL_n(\mathbb{C})$ by $\varphi^\chi_g = \chi(g)\varphi_g$. Show that φ^χ is a representation. Give an example showing that φ and φ^χ do not have to be equivalent.

Solution

1. To show that the conjugate representation is indeed a representation, we simply need to show that it is a homomorphism. To do so, let $g, h \in G$ and let's

show that $\psi_g\psi_h = \psi_{gh}$. Let $v \in V$, then

$$\begin{aligned}(\psi_g\psi_h)(v) &= \psi_g(v)\psi_h(v)\\ &= \overline{\varphi_g(v)} \cdot \overline{\varphi_h(v)}\\ &= \overline{\varphi_g(v)\varphi_h(v)}\\ &= \overline{(\varphi_g\varphi_h)(v)}\\ &= \overline{(\varphi_{gh})(v)}\\ &= (\psi_{gh})(v)\end{aligned}$$

Since it holds for all $v \in V$, then $\psi_g\psi_h = \psi_{gh}$. Therefore, ψ is a representation. An example of $\varphi \not\sim \psi$ can be obtained as follows. Take $\varphi : \mathbb{Z} \to \mathbb{C}^*$ given by $n \mapsto i^n$, then we get $\psi : \mathbb{Z} \to \mathbb{C}^*$ with $n \mapsto (-i)^n$. If we plug-in $n = 1$, we can see that $\varphi \neq \psi$. It follows that $\varphi \not\sim \psi$.

2. As for the previous part, we simply need to show that φ^χ is a homomorphism. To do so, let $g, h \in G$ and $v \in V$, then

$$\begin{aligned}(\varphi^\chi_g\varphi^\chi_h)(v) &= \varphi^\chi_g(v)\varphi^\chi_h(v)\\ &= \chi(g)\varphi_g(v)\chi(h)\varphi_h(v)\\ &= \chi(g)\chi(h)\varphi_g(v)\varphi_h(v)\\ &= \chi(gh)\varphi_{gh}(v)\\ &= \varphi^\chi_{gh}(v)\end{aligned}$$

Since it holds for all $v \in V$, then $\varphi^\chi_g\varphi^\chi_h = \varphi^\chi_{gh}$. Therefore, φ^χ is a representation.
An example of $\varphi \not\sim \varphi^\chi$ can be obtained as follows. Take $\varphi : \mathbb{Z} \to \mathbb{C}^*$ given by $n \mapsto i^n$ and $\chi : \mathbb{Z} \to \mathbb{C}^*$ given by $n \mapsto (-i)^n$, then we get $\varphi^\chi : \mathbb{Z} \to \mathbb{C}^*$ with $n \mapsto 1$. If we plug-in $n = 1$, we can see that $\varphi \neq \psi$. It follows that $\varphi \not\sim \psi$.

Exercise 3.7
Give a bijection between the unitary, degree one representations of $\mathbb{Z}$ and elements of $\mathbb{T}$.

Solution
To make things clear, consider the set $\mathrm{URep}(\mathbb{Z}, 1)$ which denotes all the unitary degree one representations of $\mathbb{Z}$. If we make no distinction between equivalent representations, then we need to find a bijection between $R = \mathrm{URep}(\mathbb{Z}, 1)/\sim$ and $\mathbb{T}$. Notice that each equivalence class E in R has a unique representative $\varphi_E : \mathbb{Z} \to \mathbb{C}^*$. Thus, consider the function $f : R \to \mathbb{T}$ defined by $f(E) = \varphi_E(1)$. Notice that f is well defined since φ_E is unique for all $E \in R$.
First, let's show that it is injective. To do so, let $E_1, E_2 \in R$ such that $f(E_1) = f(E_2)$, then by definition, $\varphi_{E_1}(1) = \varphi_{E_2}(1)$. Since φ_{E_1} and φ_{E_2} are homomorphisms, then for all $n \in \mathbb{Z}$:

$$\varphi_{E_1}(n) = [\varphi_{E_1}(1)]^n = [\varphi_{E_2}(1)]^n = \varphi_{E_2}(n)$$

Since the domains of both φ_{E_1} and φ_{E_2} is $\mathbb{Z}$, then $\varphi_{E_1} = \varphi_{E_2}$. Hence, $\varphi_{E_1} \sim \varphi_{E_2}$. This implies that E_1 and E_2 have a common representative so it follows that $E_1 =$

E_2. Thus, f is injective.
Let's now show that f is surjective. Let $e^{i\theta} \in \mathbb{T}$ and consider the map $\varphi : \mathbb{Z} \to \mathbb{C}^*$ defined by $\varphi(n) = e^{i\theta n}$. We first need to show that $\varphi \in R$. To do so, notice that it can easily be shown that φ is a homomorphism. Hence, φ is a degree one representation of $\mathbb{Z}$ that it is unitary, notice that for all $\alpha_1, \alpha_2 \in \mathbb{C}$ and $n \in \mathbb{Z}$, we have

$$\begin{aligned}\langle \varphi(n)\alpha_1, \varphi(n)\alpha_1 \rangle &= \langle e^{i\theta n}\alpha_1, e^{i\theta n}\alpha_1 \rangle \\ &= e^{i\theta n}\alpha_1 \overline{e^{i\theta n}\alpha_2} \\ &= e^{i\theta n}e^{-i\theta n}\alpha_1\overline{\alpha_2} \\ &= \alpha_1\overline{\alpha_2} \\ &= \langle \alpha_1, \alpha_2 \rangle\end{aligned}$$

Thus, φ is a unitary degree one representation, it follows that φ must be in an equivalence class $E \in R$. Moreover, φ must be the unique representation in E with codomain $\mathbb{C}^*$, i.e., $\varphi_E = \varphi$. Thus,

$$\begin{aligned}f(E) &= \varphi_E(1) \\ &= \varphi(1) \\ &= e^{i\theta}\end{aligned}$$

Thus, f is surjective since it holds for all $e^{i\theta} \in \mathbb{T}$. Therefore, f is a bijection from the unitary degree one representations up to equivalence to the set $\mathbb{T}$.

Exercise 3.8

1. Let $\varphi : G \to GL_3(\mathbb{C})$ be a representation of a finite group. Show that φ is irreducible if and only if there is no common eigenvector for the matrices φ_g with $g \in G$.

2. Give an example of a finite group G and a decomposable representation $\varphi : G \to GL_4(\mathbb{C})$ such that φ_g with $g \in G$ do not have a common eigenvector.

Solution

1. Let's prove the contrapositive instead: φ is not irreducible if and only if there is a common eigenvector for the matrices φ_g with $g \in G$.
   ($\implies$) Suppose that φ is not irreducible, then by Corollary 3.2.5, φ is decomposable. Hence, there must be non-trivial G-invariant subspaces $V_1, V_2 \leq V$ such that $V = V_1 \oplus V_2$. Since

   $$3 = \dim V = \dim V_1 \oplus V_2 = \dim V_1 + \dim V_2,$$

   then either V_1 or V_2 has dimension 1. Without loss of generality, suppose that V_1 has dimension 1, then there is a non-zero vector $u \in \mathbb{C}^3$ such that $V_1 = \mathbb{C}u$. Since V_1 is G-invariant, then for all $g \in G$ and $v \in \mathbb{C}u$, we have $\varphi_g v \in \mathbb{C}u$. In particular, if we take $v = u$, we get that for all $g \in G$, $\varphi_g u \in \mathbb{C}u$ so $\varphi_g u = \lambda_g$ where λ_g is a complex constant that depends on g. The previous statement can be restated as follows: for all $g \in G$, the vector u is an eigenvector for φ_g.

It follows that φ_g with $g \in G$ have a common eigenvector.
($\Longleftarrow$) Suppose that there is a common eigenvector for the matrices φ_g with $g \in G$. We can rephrase the previous sentence by saying that there exists a non-zero vector $u \in V$ such that for all $g \in G$, there is a constant $\lambda_g \in \mathbb{C}$ satisfying $\varphi_g u = \lambda_g u$. Consider the subspace $W = \mathbb{C}u$, let's show that W is G-invariant. To do so, let $g \in G$ and $\alpha u \in W$, then

$$\varphi_g \alpha u = \alpha \varphi_g u = \alpha \lambda_g u \in W$$

Thus, since W has dimension 1, then φ has a non-trivial proper G-invariant subspace. It follows that φ is not irreducible.

2. Consider the representation $\varphi : D_4 \to GL_2(\mathbb{C})$ described by

$$\varphi(r^k) = \begin{bmatrix} i^k & 0 \\ 0 & (-i)^k \end{bmatrix} \;,\; \varphi(sr^k) = \begin{bmatrix} 0 & (-i)^k \\ i^k & 0 \end{bmatrix}$$

We know from Exercise 1 of this chapter that the matrices φ_g with $g \in D_4$ have no common eigenvector. Consider now the representation $\psi = \varphi \oplus \varphi$. Suppose that the matrices ψ_g with $g \in D_4$ have a common eigenvector, then there exists a non-zero vector $u \in \mathbb{C}^2 \times \mathbb{C}^2$ such that for all $g \in D_4$, there exists a constant $\lambda_g \in \mathbb{C}$ such that $\psi_g u = \lambda_g u$. If we write u as (u_1, u_2) where both u_1 and u_2 are two vectors in $\mathbb{C}^2$, then one of u_1 and u_2 must be non-zero since u is non-zero. Suppose without loss of generality that u_1 is non-zero. Then we get that for all $g \in D_4$, there is a constant λ_g such that

$$\begin{aligned} \psi_g u = \lambda_g u &\implies (\varphi_g \oplus \varphi_g)(u_1, u_2) = \lambda_g (u_1, u_2) \\ &\implies (\varphi_g u_1, \varphi_g u_2) = (\lambda_g u_1, \lambda_g u_2) \\ &\implies \varphi_g u_1 = \lambda_g u_1 \end{aligned}$$

In other words, there is a non-zero vector $u_1 \in \mathbb{C}^2$ such that for all $g \in D_4$ there is a $\lambda_g \in \mathbb{C}$ satisfying $\varphi_g u_1 = \lambda_g u_1$. But this is a contradiction since it would imply that the matrices φ_g with $g \in D_4$ have a common eigenvector. Thus, by contradiction, the matrices ψ_g with $g \in D_4$ have no common eigenvector. Notice that ψ is equivalent to a representation $\psi' : D_4 \to GL_4(\mathbb{C})$ so the same conclusion holds for this new representation.

Chapter 4

Character Theory and the Orthogonality Relations

Exercise 4.1
TODO